

Proof of a conjecture of Kenyon and Wilson on semicontiguous minors

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Abstract

Kenyon and Wilson showed how to test if a circular planar electrical network with n nodes is well-connected by checking the positivity of $\binom{n}{2}$ central minors of the response matrix ([arXiv:1411.7425](https://arxiv.org/abs/1411.7425)). Their test is based on the fact that any contiguous minor of a matrix can be expressed as a Laurent polynomial in the central minors. Interestingly, the Laurent polynomial is the generating function of domino tilings of a weighted Aztec diamond. They conjectured that any semicontiguous minor can also be written in terms of domino tilings of a region on the square lattice. In this paper we present a proof of the conjecture.

Keywords: perfect matchings, domino tilings, dual graph, graphical condensation, electrical networks, response matrix, Aztec diamonds.

1 Introduction

The study of the electrical networks comes from classical physics with the work of Ohm and Kirchhoff more than 100 years ago. The *circular planar electrical networks* were first studied systematically by Colin de Verdière [[Col94](#)] and Curtis, Ingerman, Moores, and Morrow [[CIM](#), [CMM](#)]. Recently, a number of new properties of the circular planar electrical networks have been discovered (see e.g. [[ALT](#), [KW10](#), [KW14](#), [Lam](#), [LP](#), [Yi](#)]).

A *circular planar electrical network* (or simply *network* in this paper) is a finite graph $G = (V, E)$ embedded on a disk with a set of distinguished vertices $N \subseteq V$ on the circle, called *nodes*, and a *conductance function* $wt : E \rightarrow \mathbb{R}^+$ (see Figure 1.1 for an example).

Arrange the indices $1, 2, \dots, n$ of a $n \times n$ matrix $M = (m_{i,j})_{1 \leq i, j \leq n}$ in counter-clockwise order around the circle. Assume that $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_\ell\}$ are two sets of indices so that a_1, a_2, \dots, a_k and $b_\ell, b_{\ell-1}, \dots, b_1$ are in counter-clockwise order around the circle. We denote by M_A^B the submatrix $(m_{a_i, b_j})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}}$ of M . In the case $k = \ell$, we call the

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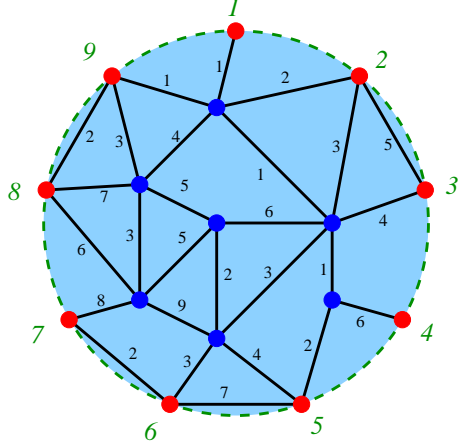


Figure 1.1: A circular planar electrical network with 9 nodes.

pair (A, B) a *circular pair* of M and the determinant $\det M_A^B$ a *circular minor*¹ of M . If A and B are non-interlaced around the circle, we call the later minor $\det M_A^B$ a *non-interlaced circular minor*.

Associated with a network with n nodes is a *response matrix* $\Lambda = (\lambda_{i,j})_{1 \leq i,j \leq n}$ that measures the response of the network to potential applied at the nodes. In particular, $-\lambda_{i,j}$ is the current that would flow into node j if node i is set to one volt and the remaining nodes are set to zero volts. It has been shown that a matrix M is the response matrix of a network if and only if it is symmetric with row and column sums equal zero, and each non-interlaced circular minor $\det M_A^B$ is non-negative (see Theorem 4 in [CIM]).

A network is called *well-connected* if for any two non-interlaced sets of k nodes A and B , there are k pairwise vertex-disjoint paths in G connecting the nodes in A to the nodes in B . A number of equivalent definitions of the well-connected networks were given in [Col94]. A network is well-connected if and only if all non-interlaced circular minors of the response matrix are positive.

A *contiguous minor* of a matrix M is a circular minor of the form

$$\text{CON}_{a,b,y}(M) := \det M_A^B, \quad (1.1)$$

where $A = \{a, a+1, \dots, a+y-1\}$ and $B = \{b+y-1, \dots, b+1, b\}$, and where the indices are interpreted modulo n (i.e. the row indices and the column indices are contiguous on the circle). The *central minor* $\text{CM}_{x,y}(M)$ of M is defined to be the contiguous minor $\text{CON}_{a,b,y}(M)$ with

$$a = \left\lfloor \frac{x-y}{2} \right\rfloor \text{ and } b = \left\lfloor \frac{x-y+n-(n-1 \bmod 2)}{2} \right\rfloor.$$

The central minor was defined implicitly in [CIM]. One readily sees that the parameter x is naturally interpreted in modulo $2n$ (since increasing x by 2 is equivalent to cyclically shifting the indices 1 unit counter-clockwise). The parameter y ranges from 0 to n .

If $1 \leq x \leq n$, $1 \leq y < n/2$ or $y = n/2$ and $x+y$ is odd, then we call $\text{CM}_{x,y}(M)$ is a *small central minor*.

¹In this paper, we refer to the determinants of submatrices as minors.

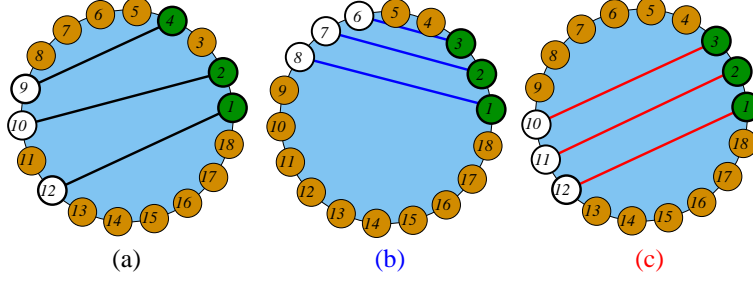


Figure 1.2: Three circular minors of a 18×18 matrix M : (a) $M_{1,2,4}^{12,10,9}$, (b) $\text{CON}_{1,6,3}(M)$, and (c) $\text{CM}_{6,3}(M)$.

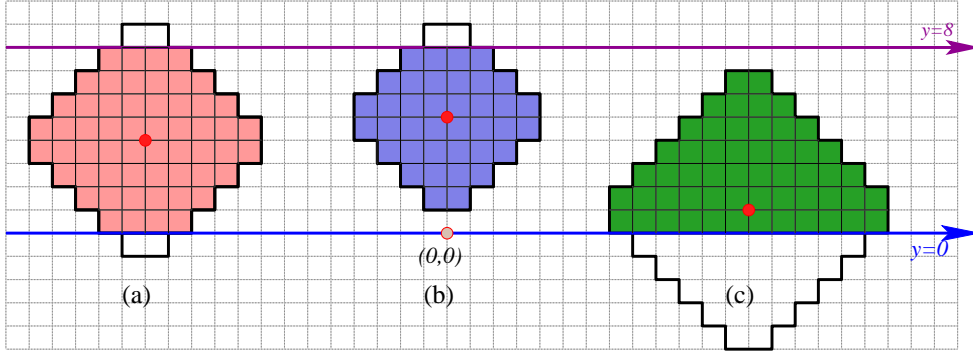


Figure 1.3: The truncated Aztec diamonds: (a) $\text{TAD}_{-13,4}^{5,8}$, (b) $\text{TAD}_{0,5}^{4,8}$, (c) $\text{TAD}_{13,1}^{6,8}$.

In this paper, a circular minor $\det M_A^B$ is usually represented by a circular diagram with k chords connecting a_i to b_i , for $i = 1, 2, \dots, k$ (where $|A| = |B| = k$). In this representation, the central minors have their chords as centrally located as possible (plus or minus a rounding error). Figure 1.2 illustrates three circular minor of a 18×18 matrix M : (a) a non-contiguous minor, (b) a contiguous minor, which is not a central minor, and (c) a central minor.

There are $\binom{n}{2}$ small central minors, whether n is even or odd. Kenyon and Wilson [KW14] showed how to test the well-connectivity of a network by checking the positivity of the $\binom{n}{2}$ small central minors of the response matrix. This means that the positivity of the central minors implies the positivity of all circular minors.

The *Aztec diamond* AD_{x_0, y_0}^h of order h with the center located at the lattice point (x_0, y_0) in the grid \mathbb{Z}^2 is the region consisting of unit squares inside the contour $|x - x_0| + |y - y_0| = h + 1$. It has been proven that there are $2^{h(h+1)/2}$ different ways to cover an Aztec diamond of order h by dominoes so that there are no gaps or overlaps [EKLP1, EKLP2]; and such coverings are called *domino tilings* of the Aztec diamond. The *truncated Aztec diamond* $\text{TAD}_{x_0, y_0}^{h, n}$ is defined to be the portion of the Aztec diamond AD_{x_0, y_0}^h between the lines $y = 0$ and $y = n$ (see Figure 1.3 for several examples). We notice that $\text{TAD}_{x_0, y_0}^{h, n} \equiv \text{AD}_{x_0, y_0}^h$ if $h \leq y_0 \leq n - h$.

Besides the Aztec diamonds, we are also interested their natural generalizations, the *Aztec rectangles*. The Aztec rectangle of size 3×6 is illustrated in Figure 1.4(a); the Aztec rectangle of size 4×6 is shown in Figure 1.4(b). The lattice point (x_0, y_0) is called the *center* of the Aztec rectangle if the line $x = x_0$ passes through the middle point of the top length-2 step of the boundary, and the line $y = y_0$ passes through the middle point of the length-2 vertical step

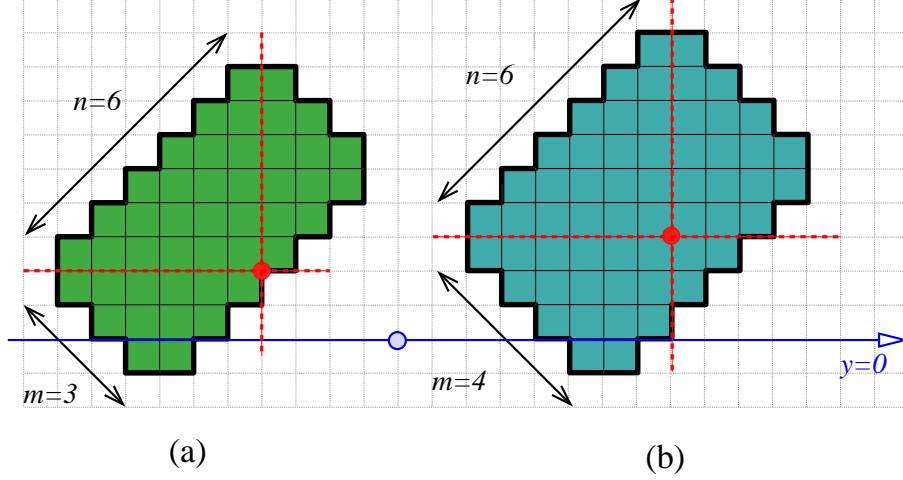


Figure 1.4: The Aztec rectangles.

on the left of the boundary (see the dots in Figure 1.4). Denote by $\text{AR}_{x_0, y_0}^{m, n}$ the Aztec rectangle of size $m \times n$ with the center at (x_0, y_0) .

Given a $n \times n$ matrix M . For each lattice point (x, y) , we define $v_{x, y} := \text{CM}_{x, y}$ if $0 \leq y \leq n$, and $v_{x, y} := 1$ otherwise. We assign to each domino a weight $\frac{1}{v_{x_1, y_1} v_{x_2, y_2}}$, where $v_{x, y}$ denotes the minor $\text{CM}_{x, y}(M)$, and where (x_1, y_1) and (x_2, y_2) are the middle points of the long sides of the domino. In particular, the horizontal domino consisting of the unit squares centered at $(x + \frac{1}{2}, y + \frac{1}{2})$ and $(x - \frac{1}{2}, y + \frac{1}{2})$ is weighted by $\frac{1}{v_{x, y} v_{x, y+1}}$; analogously, the vertical domino covering the unit squares centered at $(x + \frac{1}{2}, y + \frac{1}{2})$ and $(x + \frac{1}{2}, y - \frac{1}{2})$ has weight $\frac{1}{v_{x, y} v_{x+1, y}}$. The *weight* of a domino tiling of a R is the product of weights of all dominoes in the tiling. The weight $W(R)$ of a region R is the sum of weights of all the domino tilings of R (if R does not have any domino tiling, then $W(R) := 0$). Our weight assignment here can be viewed as the ‘dual’ of Speyer’s weight assignment in [Spe].

The *covering monomial* $F(R)$ of a non-empty region R is defined to be the product $\prod_{x, y} v_{x, y}$ taken over all the lattice points (x, y) inside R or on the boundary of R , except for 90° -corners (the lattice points (x, y) are illustrated by the dots in Figure 1.5). The zero-order Aztec diamond AD_{x_0, y_0}^0 is a formal empty region, which has the weight $W(\text{AD}_{x_0, y_0}^0) := 1$ and the covering monomial $F(\text{AD}_{x_0, y_0}^0) := v_{x_0, y_0}$.

To a region R , we associate a Laurent polynomial $P(R) := F(R) W(R)$ in the variables $v_{x, y}$ ’s. We call $P(R)$ the *tiling polynomial* of the region R . Kenyon and Wilson [KW14] proved that any contiguous minor can be written as the tiling polynomial of a truncated Aztec diamond.

Theorem 1.1 (Kenyon and Wilson [KW14]). *Let $\text{CON}_{a, b, y}(M)$ be a contiguous minor of a matrix M . Assume that h is the integer closest to 0 so that $\text{CON}_{a, b+h, y}(M)$ is the central minor $\text{CM}_{x, y}(M)$. Then $\text{CON}_{a, b, y}(M) = P\left(\text{TAD}_{x-h, y}^{[h], n}\right)$.*

For example, let M be a 13×13 matrix, then the contiguous minor $\text{CM}_{1, 5, 2}(M)$ is expressed as the tiling polynomial $P(\text{TAD}_{2, 2}^{2, 13})$ (shown in Figure 1.6). We refer the reader to [KW14, pp. 17–18] for more examples.

A *semicontiguous minor* is a minor of the form $\det M_A^B$, where *exactly one* of A and B is

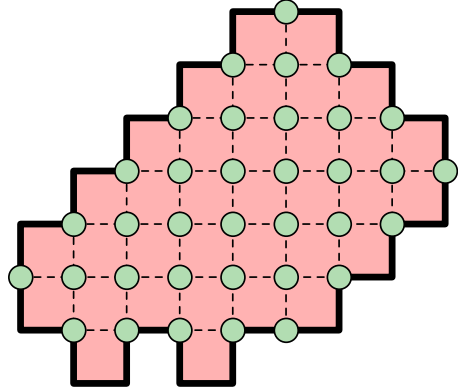


Figure 1.5: Illustration of the definition of covering monomial.

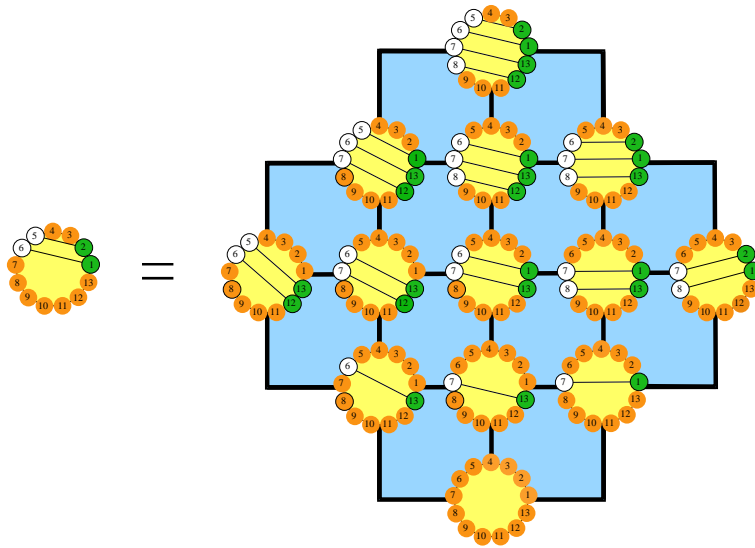


Figure 1.6: The correspondence between contiguous minors and truncated Aztec diamonds.

contiguous. Kenyon and Wilson conjectured in Section 4.3 of [KW14] that

Conjecture 1.2 (Kenyon and Wilson). *Any semicontiguous minor can be written as the tiling polynomial $P(R)$ of some region R on the square lattice.*

The goal of this paper is to prove this conjecture. Our proof uses a variation of *Dodgson condensation* (or *Desnanot-Jacobi identity*, see e.g. [Dod] and [Mui, pp. 136–149]) due to Kenyon and Wilson [KW14] and a powerful method in enumeration of tilings and perfect matchings, *Kuo condensation* [Kuo04]. We refer the reader to e.g. [Ciu, YYZ, YZ, Ful, Spe] for various aspects and generalizations of the method. More recent applications of Kuo condensation can be found in e.g. [CF16, CF15, CL, Lai1, Lai2, Lai3, Lai4, LMNT, KW14].

The rest of this paper is organized as follows. Our main results are presented in Section 2. In Section 3, we show the particular versions of Dodgson and Kuo condensations, which will be employed in our proof. The proof of our main results will be shown in Section 4. Finally, we conclude the paper by posing an open question for the case of general circular minors.

2 The main results

In this section, we will describe carefully the structure of the regions corresponding to the semicontiguous minors.

Consider a circular minor M_A^B of a $n \times n$ matrix M , where at least one of A and B is contiguous. We consider first the case when A is contiguous, then B may be not contiguous. Assume that B is partitioned into s contiguous sets B_1, B_2, \dots, B_s (in counter-clockwise order around the circle). Assume in addition that $|B_i| = k_i > 0$, and that there are t_i indices ($t_i > 0$), which are not in B , staying between B_i and B_{i+1} . We call the sets of indices separating two consecutive B_i 's the *gaps* of B . Figure 2.1(a) shows an example of the semicontiguous minor with three gaps in B for the case $n = 60$, $s = 4$, $k_1 = 3$, $k_2 = 4$, $k_3 = 3$, $k_4 = 2$, $t_1 = t_2 = t_3 = 2$ (the indices of each set B_i are represented by adjacent nodes of the same color). Denote by $k := k_1 + \dots + k_s$ and $t := t_1 + \dots + t_{s-1}$. By definition we always have $k + t \leq n$. We also assume that the first index in A is a (i.e., $A = \{a, a+1, \dots, a+k-1\}$), and the first index in B is b . We denote by $\text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1}) = \text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1})(M)^2$ for this minor. We notice that when $s = 1$, $\text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ is the contiguous minor $\text{CON}_{a,b,k_1}(M)$, and when $s \geq 2$, it is a semicontiguous minor.

Let A_1 be the set consisting of the last k_1 indices in A , then $\det M_{A_1}^{B_1}$ is a circular minor of M . We assume that $\text{TAD}_{x-h,k_1}^{[h],n}$ is the truncated Aztec diamond corresponding to $\det M_{A_1}^{B_1}$ as in Theorem 1.1, i.e. h the integer closest to zero so that the contiguous minor $M_{A_1}^{B_1+h}$ is the central minor $\text{CM}_{x,k_1}(M)$. Here $B_1 + h$ is the set obtained from B by translating it h units counter-clockwise.

Remark 2.1. Fix the index a , and let the index b run along the circle. There are two cases in which $h = 0$, i.e. in which $M_{A_1}^{B_1}$ is a central minor. We use the notations 0^+ and 0^- to distinguish these zero values of h . More precise, we define $h := 0^+$ if $b = \lfloor \frac{n-1}{2} \rfloor + a + k - k_1$, and $h := 0^-$ if $b = \lfloor \frac{n-1}{2} \rfloor + a + k - k_1 + 1$. We say that $h \geq 0^+$ if $h = 0^+$ or $h \geq 1$, and that $h \leq 0^-$ if $h = 0^-$ or $h \leq -1$.

²From now on, if the matrix M is given, we usually drop the parameter M in the notation of the SM-minors.

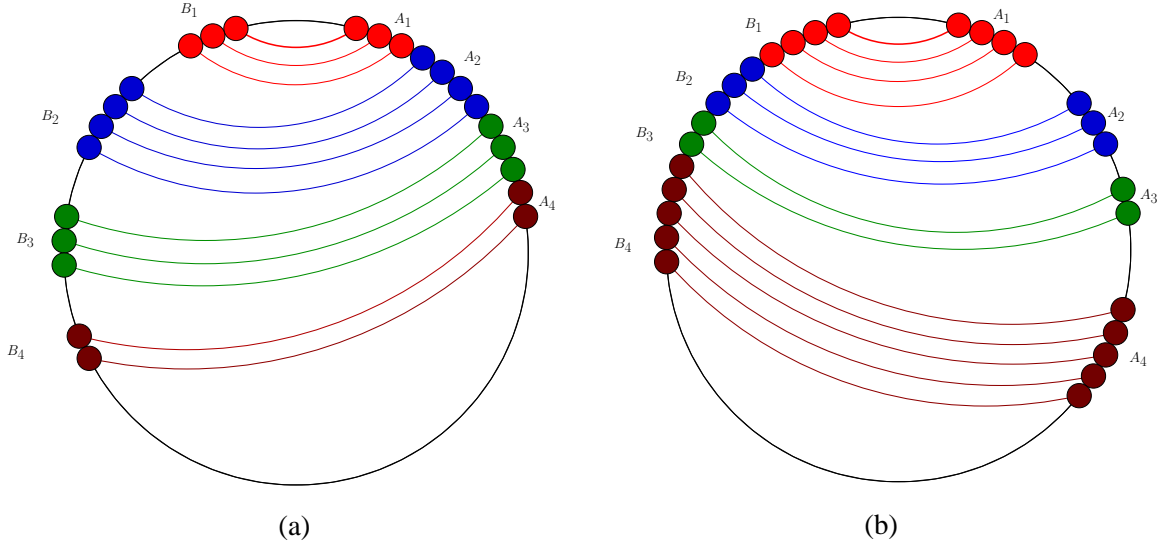


Figure 2.1: (a) The semicontiguous minor with gaps in B . (b) The semicontiguous minor with gaps in A .

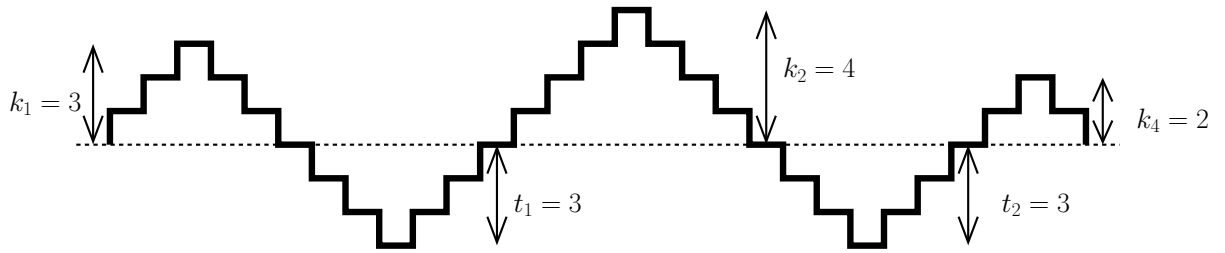


Figure 2.2: The zigzag path $\mathcal{P}(3, 4, 2; 3, 3)$.

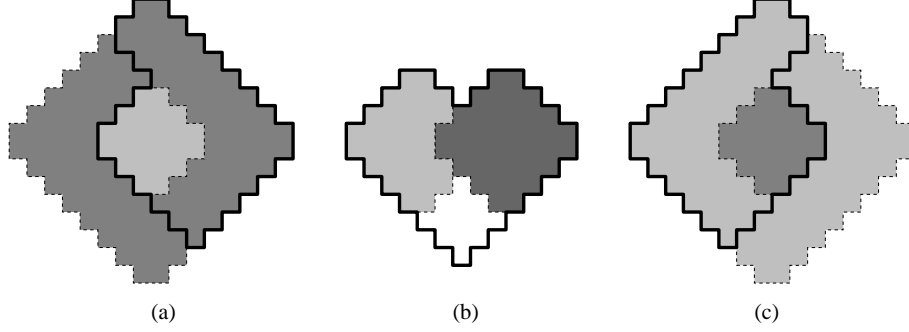


Figure 2.3: The L -sum of two overlapped Aztec diamonds.

We define a zigzag path $\mathcal{P} := \mathcal{P}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ consisting of north and east steps, and starting and ending on the line $y = 0$ as follows. \mathcal{P} starts with a peak of height k_1 , and contains alternatively a valley of depth t_i and a peak of height k_{i+1} , for $i = 1, 2, \dots, s-1$ (see Figure 2.2 for an example). We use the notation $\mathcal{P}^+ := \mathcal{P}^+(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ (resp., $\mathcal{P}^- := \mathcal{P}^-(k_1, \dots, k_s; t_1, \dots, t_{s-1})$) for the infinite lattice path obtained from \mathcal{P} by extending horizontally to $+\infty$ (resp., $-\infty$) from the right (resp., left) endpoint.

For any two overlapping Aztec diamonds $\text{AD}_1 := \text{AD}_{x_1,0}^{h_1}$ and $\text{AD}_2 := \text{AD}_{x_2,0}^{h_2}$, we define the L -sum $\text{AD}_1 \oplus_L \text{AD}_2$ as in Figure 2.3, where AD_1 is the light shaded diamond and AD_2 is the dark shaded one. More precise, if AD_1 stays inside AD_2 , then $\text{AD}_1 \oplus_L \text{AD}_2$ is the L -shaped region restricted by the bold contour as in Figure 2.3(a); if AD_2 stays inside AD_1 , then $\text{AD}_1 \oplus_L \text{AD}_2$ is the L -shaped region restricted by the bold contour as in Figure 2.3(c); finally if the two diamonds do not contain each other, $\text{AD}_1 \oplus_L \text{AD}_2$ is the V -shaped region as in Figure 2.3(b).

Next, we define a family of regions $\mathcal{Q} = \mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ as follows.

If $s = 1$, then $\mathcal{Q} = \mathcal{Q}_{x,h}(k_1; \emptyset) := \text{TAD}_{x-h,k_1}^{[h],n}$, the truncated Aztec diamond corresponding to the contiguous minor $M_{A_1}^{B_1}$ as in Theorem 1.1. When $s \geq 2$, there are three types of the region \mathcal{Q} as follows:

Type 1. $t < h - k$. Removing all unit squares in the Aztec rectangle $\text{AR}_{x-h,0}^{h+k_1, h-k+k_1}$, which are below the zigzag path $\mathcal{P}^+ := \mathcal{P}^+(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ with the right endpoint at the right corner of the Aztec rectangle (see the bold zigzag paths on the left pictures in Figure 2.4), we get the region $\mathcal{H} = \mathcal{H}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ (shown by the shaded region on the left pictures in Figure 2.4). Finally, we define \mathcal{Q} to be the portion of \mathcal{H} between the lines $y = 0$ and $y = n$.

Type 2. $h \geq 0^+$ and $t \geq h - k$. The region \mathcal{Q} is obtained by applying the above double-trimming process to the region $\mathcal{R} := \text{AD}_1 \oplus_L \text{AD}_2$, where $\text{AD}_1 := \text{AD}_{x-h,0}^{h+k_1}$ and $\text{AD}_2 := \text{AD}_{x-h+t,0}^{2k+t-h-k_1-1}$ (instead of the Aztec rectangle $\text{AR}_{x-h,0}^{h+k_1, h-k+k_1}$ as in type 1). In particular, \mathcal{H} is the portion of \mathcal{R} above the zigzag path \mathcal{P}^+ ; and our region \mathcal{Q} is obtained from \mathcal{H} by truncating the part below the line $y = 0$ and the part above the line $y = n$. See Figure 2.5 for three examples corresponding to three possible shapes of the region \mathcal{R} as in Figure 2.3.

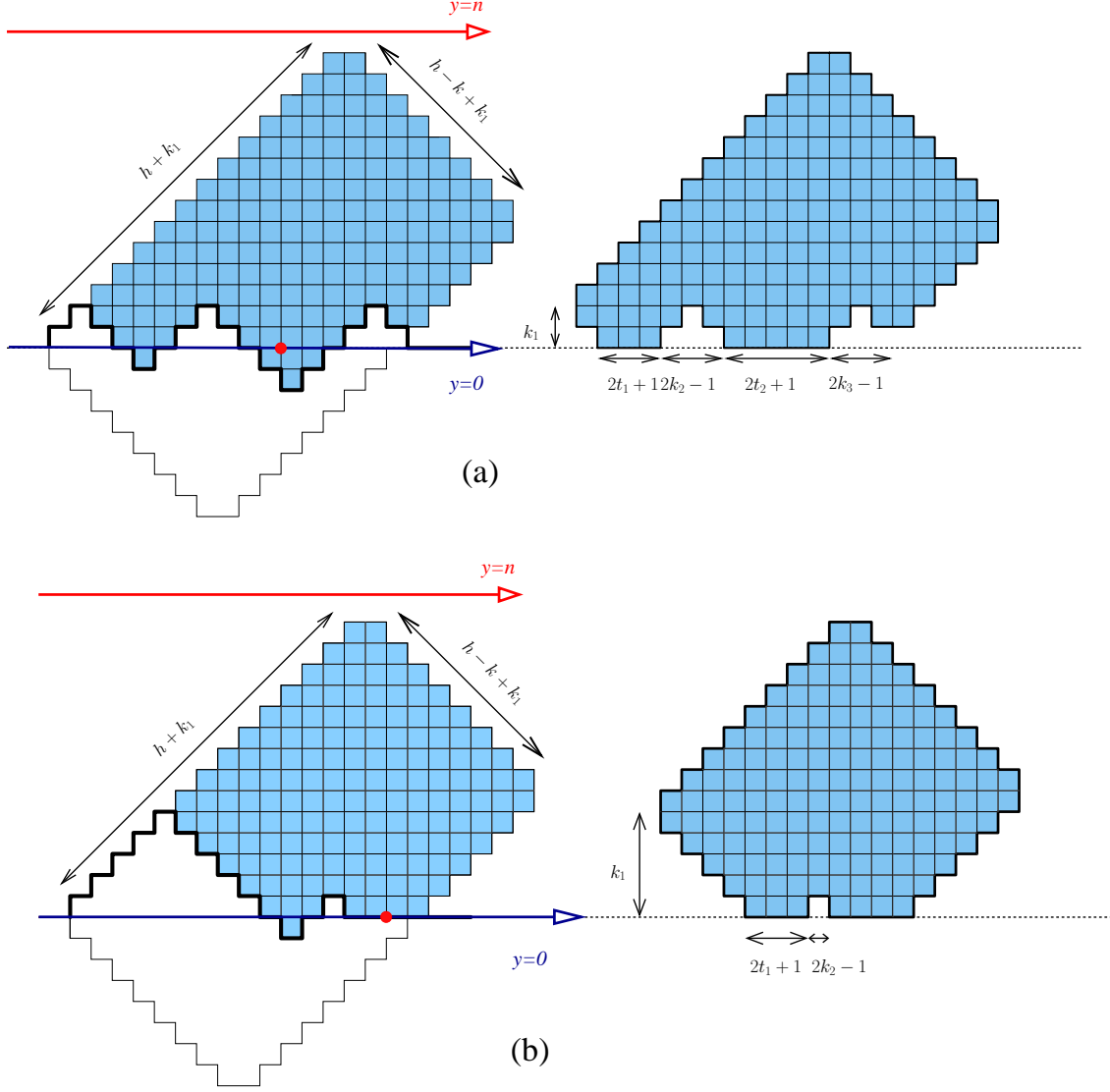


Figure 2.4: Obtaining the type-1 region $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ by truncating $\mathcal{H}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$: (a) The case when $s = 3$, $k_1 = k_2 = k_3 = 2$, $t_1 = 1$, $t_2 = 2$, $x = 15$, $h = 12$, (b) The case when $s = 2$, $k_1 = 4$, $k_2 = 1$, $t_1 = 1$, $x = 8$, $h = 9$. The dots indicate the origin $(0, 0)$.

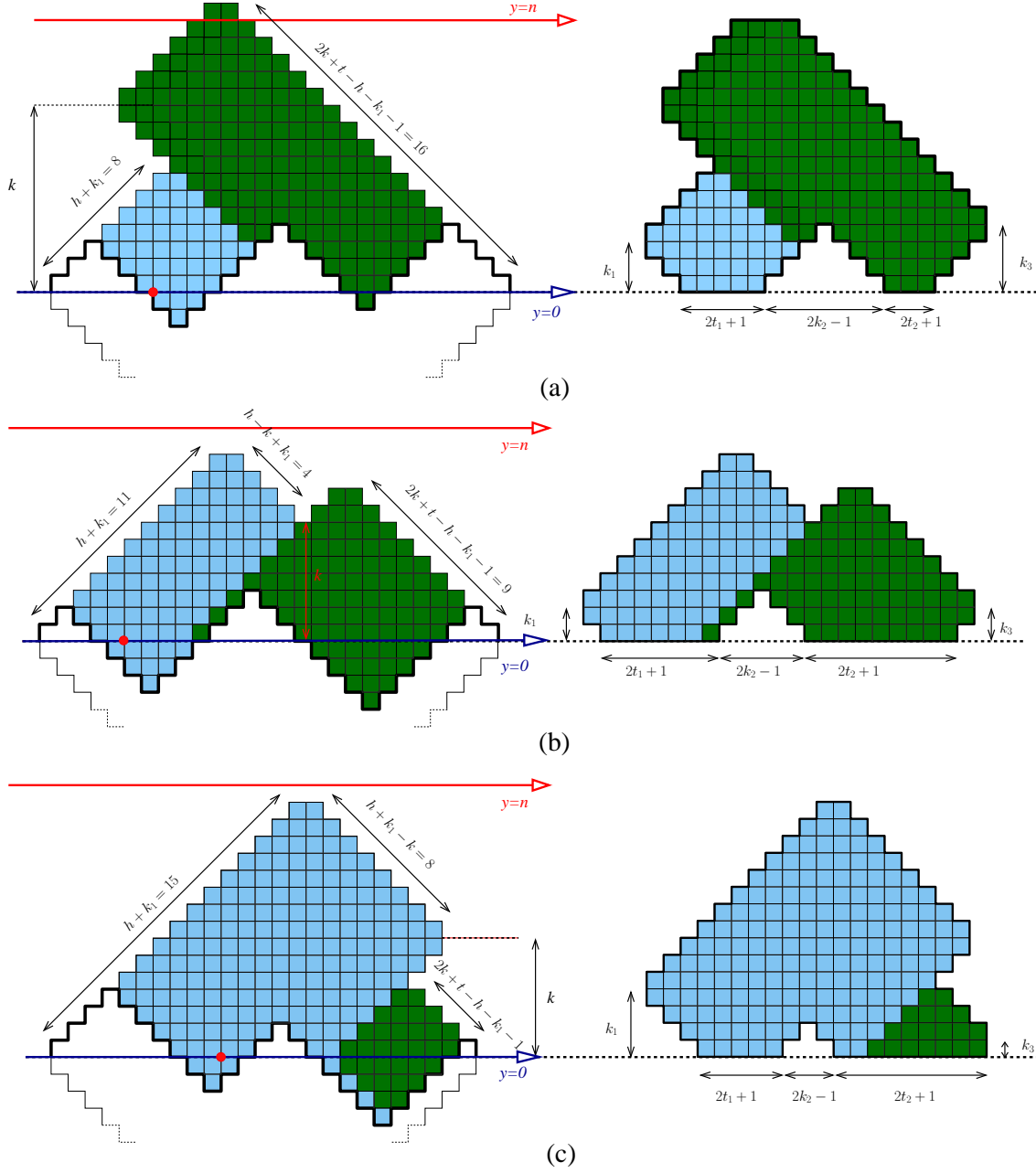
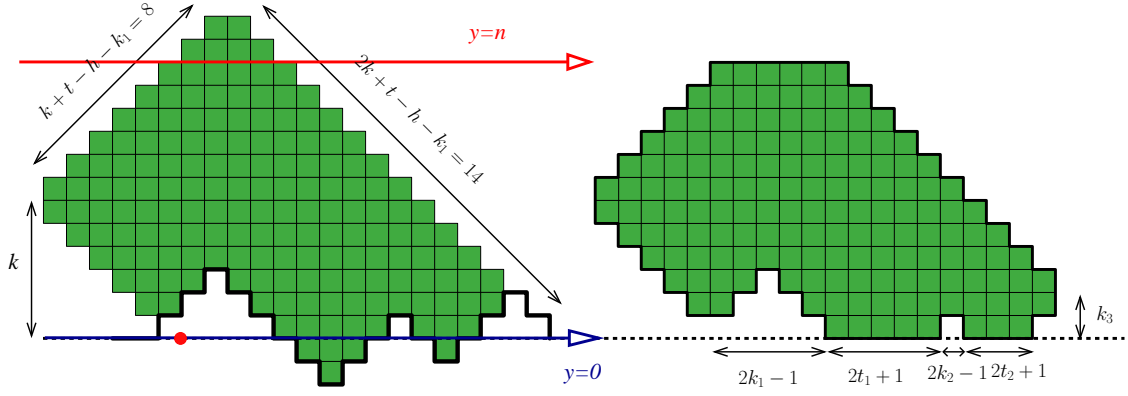
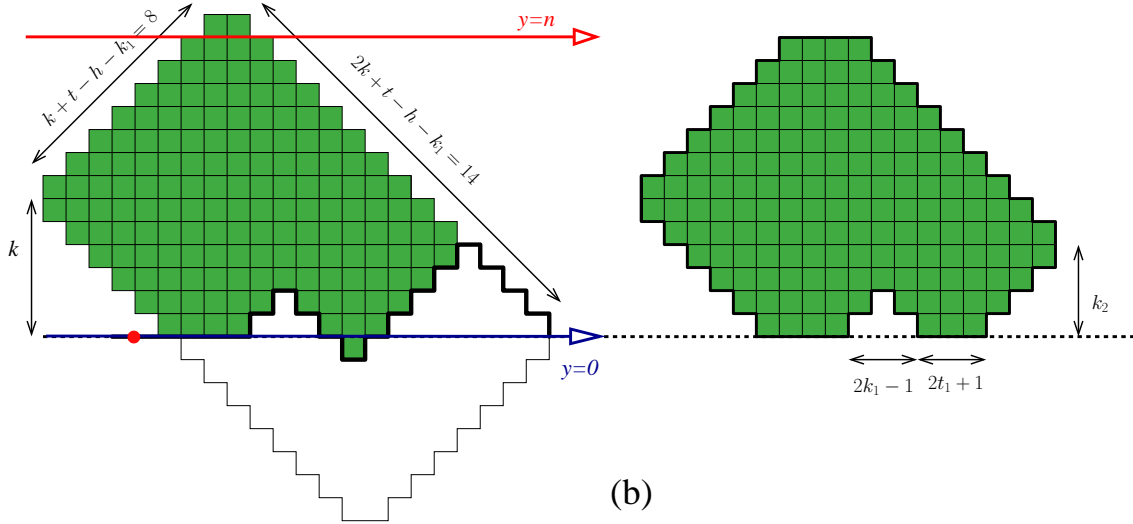


Figure 2.5: Obtaining the region $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ of Type 2 from $\mathcal{H}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$: (a) for $s = 3$, $k_1 = 3$, $k_2 = 4$, $k_3 = 4$, $t_1 = 2$, $t_2 = 1$, $x = 7$, $h = 4$, (b) for $s = 3$, $k_1 = 2$, $k_2 = 3$, $k_3 = 2$, $t_1 = 3$, $t_2 = 4$, $x = 15$, $h = 9$, and (c) for $s = 3$, $k_1 = 4$, $k_2 = 2$, $k_3 = 1$, $t_1 = 2$, $t_2 = 4$, $x = 16$, $h = 11$. The portion of AD_1 has the light shading, and the portion of AD_2 has the dark shading.



(a)



(b)

Figure 2.6: Obtaining the region $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ of Type 3 from $\mathcal{H}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$: (a) The example for $s = 3$, $k_1 = 3$, $k_2 = 1$, $k_3 = 2$, $t_1 = 2$, $t_2 = 1$, $x = 0$, $h = -2$. (b) The example for $s = 2$, $k_1 = 2$, $k_2 = 4$, $t_1 = 1$, $x = 1$, $h = -3$.

Type 3. $h \leq 0^-$. We start with the Aztec rectangle $\text{AR}_{x-h+t,0}^{k+t-h-k_1, 2k+t-h-k_1}$, then remove all unit squares below the zigzag path \mathcal{P}^- with the *left* endpoint at the *left* corner of the rectangle, and truncate the part below the line $y = 0$ and the part above the line $y = n$ from the resulting region. Two examples of the region \mathcal{Q} in this case are shown in Figure 2.6.

Remark 2.2. We notice that in Type 1, the top of the region \mathcal{H} is always below the line $y = n$ (as $h + k_1 < h + k + t < 2h \leq n$, since we are assuming that $t < h - k$, and $|h| \leq n/2$ by definition). However, in Types 2 and 3, the top of the region \mathcal{H} may stay above the line $y = n$.

Theorem 2.3. *Assume that $s, k_1, \dots, k_s, t_1, \dots, t_{s-1}$ are positive integers, and that M is a square matrix. Assume in addition that $\text{TAD}_{x-h, k_1}^{[h], n}$ is the truncated Aztec diamond corresponding to the contiguous minor $M_{A_1}^{B_1}$ as in Theorem 1.1. Then*

$$\text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1}) = \text{P}(\mathcal{Q}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1})). \quad (2.1)$$

Next, we describe the region corresponding to the semicontiguous minor M_A^B , where B is contiguous.

We now assume the decomposition $A = \bigcup_{i=1}^s A_i$, where A_i 's are contiguous index sets appearing in *clockwise* order around the circle. Assume in addition that $|A_i| = k_i > 0$, and that the size of the gap between A_i and A_{i+1} is $t_i > 0$ (see Figure 2.1(b) for an example for $n = 60$, $s = 4$, $k_1 = 4$, $k_2 = 3$, $k_3 = 2$, $k_4 = 5$, $t_1 = 2$, $t_2 = 1$, $t_3 = 3$). We also assume that a and b are respectively the first indices of A and B as usual (i.e., we have now $B = \{b, b+1, \dots, b+k-1\}$). Denote by $\overline{\text{SM}}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1}) = \overline{\text{SM}}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1})(M)$ this minor. We also note that $\overline{\text{SM}}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ is contiguous when $s = 1$, and semicontiguous when $s \geq 2$.

Intuitively, the region $\overline{\mathcal{Q}}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ corresponding to the above $\overline{\text{SM}}$ -minor is obtained from the region $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ by reflecting it over a vertical line and translating horizontally. In particular, the region $\overline{\mathcal{Q}}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ is obtained by truncating the part below the line $y = 0$ and the part above the line $y = n$ from the region $\overline{\mathcal{H}} = \overline{\mathcal{H}}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ that is defined as follows. If $t < h - k$, $\overline{\mathcal{H}}$ is obtained from the Aztec rectangle $\text{AR}_{x-h,0}^{h-h+k_1, h+k_1}$ by removing all unit squares below the zigzag path $\overline{\mathcal{P}}^-$, where $\overline{\mathcal{P}} := \mathcal{P}(k_s, \dots, k_1; t_{s-1}, \dots, t_1)$, with the left endpoint at the left corner of the Aztec rectangle (see the left picture on the top row in Figure 2.7); if $h \geq 0^+$ and $t \geq h - k$, then $\overline{\mathcal{H}}$ is obtained by the same process for the L -sum $\overline{\mathcal{R}} := \text{AD}_{x-h,0}^{h+k_1} \oplus_L \text{AD}_{x-h-t,0}^{2k+t-h-k_1-1}$ (illustrated by the left picture on the middle row in Figure 2.7); finally if $h \leq 0^-$, then $\overline{\mathcal{H}}$ is obtained from $\text{AR}_{x-h-t,0}^{k+t-k_1-h, k+t-h}$ by removing all the unit squares below the zigzag line $\overline{\mathcal{P}}^+$ with the right endpoint at the right corner of the Aztec rectangle (pictured by the left picture on the bottom row in Figure 2.7).

Similar to Theorem 2.3, we have the following theorem for $\overline{\mathcal{Q}}$ -type regions.

Theorem 2.4. *Assume that $s, k_1, \dots, k_s, t_1, \dots, t_{s-1}$ are positive integers, and that M is a square matrix. Assume in addition that $\text{TAD}_{x-h, k_1}^{[h], n}$ is the truncated Aztec diamond corresponding to the contiguous minor $M_{A_1}^{B_1}$ as in Theorem 1.1. Then*

$$\overline{\text{SM}}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1}) = \text{P}(\overline{\mathcal{Q}}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})). \quad (2.2)$$

One readily sees that Theorems 2.3 and 2.4 verify Conjecture 1.2.

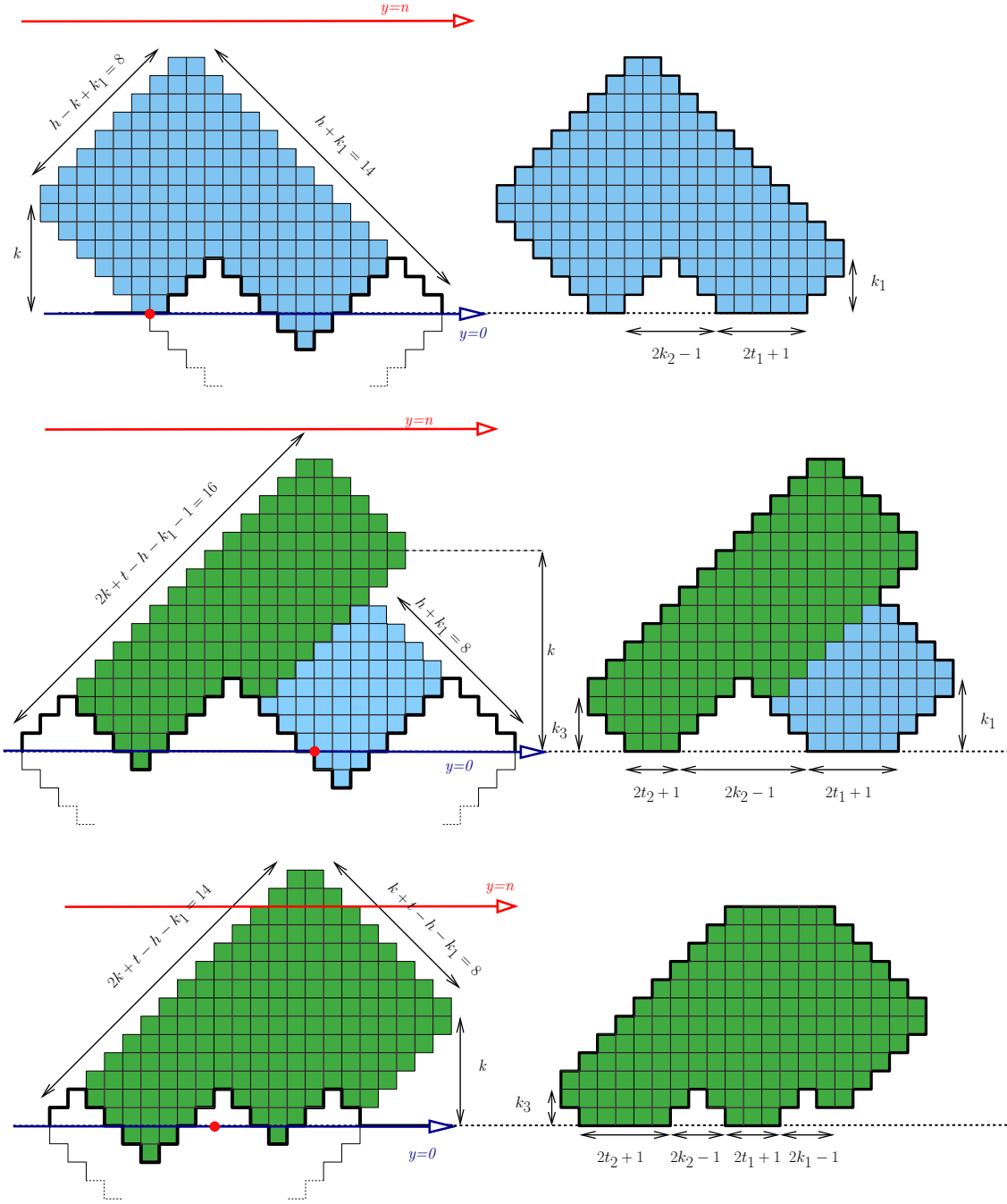


Figure 2.7: Obtaining the region $\overline{\mathcal{Q}}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ from the region $\overline{\mathcal{H}}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$: (a) The case when $s = 2, k_1 = 3, k_2 = 3, t_1 = 2, x = 13, h = 11$. (b) The example for $s = 3, k_1 = 4, k_2 = 4, k_3 = 3, t_1 = 2, t_2 = 1, x = 7, h = 4$. (c) The example for $s = 3, k_1 = k_2 = k_3 = 2, t_1 = 1, t_2 = 2, x = 4, h = -1$.

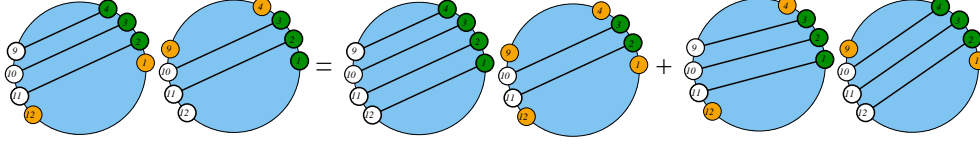


Figure 3.1: Illustration of Dodgson condensation for $M = \Omega_{1,2,3,4}^{12,11,10,9}$, $a = 1$, $b = 4$, $c = 12$, and $d = 9$.

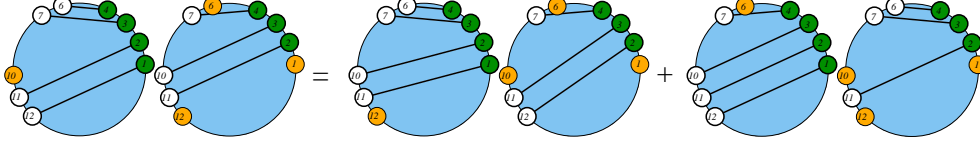


Figure 3.2: Illustration of jaw move for $M = \Omega_{1,2,3,4}^{12,11,10,7,6}$, $d = 12$, $e = 10$, $f = 6$, and $g = 1$.

3 Dodgson Condensation and Kuo Condensation

Given a matrix M , we denote by $M_{\widehat{a_1, \dots, a_k} \widehat{b_1, \dots, b_l}}$ the matrix obtained from M by removing the rows a_1, a_2, \dots, a_k and the columns b_1, b_2, \dots, b_l . We employ the following Dodgson condensation [Dod] and its variation in our proof.

Lemma 3.1 (Dodgson condensation). *Let M be a $n \times n$ matrix. Then*

$$\det M_{\widehat{a}}^{\widehat{c}} \det M_{\widehat{b}}^{\widehat{d}} = \det M \det M_{\widehat{a,b}}^{\widehat{c,d}} + \det M_{\widehat{b}}^{\widehat{c}} \det M_{\widehat{a}}^{\widehat{d}}, \quad (3.1)$$

where the indices a, b, d, c appear in counter-clockwise order around the circle. See Figure 3.1 for an example.

The following variation of Dodgson condensation, called “jaw move”, is due to Kenyon and Wilson (see the proof of Theorem 7 in [KW14]).

Lemma 3.2 (Jaw Move). *Let M be a $n \times (n + 1)$ matrix. Then*

$$\det M_{\widehat{g}}^{\widehat{e}} \det M_{\widehat{f}}^{\widehat{d,e}} = \det M_{\widehat{g}}^{\widehat{d}} \det M_{\widehat{f}}^{\widehat{e,f}} + \det M_{\widehat{f}}^{\widehat{e}} \det M_{\widehat{g}}^{\widehat{d,e}}, \quad (3.2)$$

where the indices g, f, e, d appear in counter-clockwise order around the circle. The jaw move is illustrated in Figure 3.2.

Let Ω be a matrix, then the condensations in the above lemmas are illustrated in Figures 3.1 and 3.2.

A *perfect matching* of a (simple, finite) graph $G = (V, E)$ is a collection of disjoint edges in E which cover all vertex set V . Similar to the case of tilings, we define the weight $W(G)$ of the graph G to be the sum of the weights of all perfect matchings of G , where the weight of a perfect matching is the product of weights of its edges.

The *dual graph* of a region R on the square lattice is the graph whose vertices are the unit squares in R and whose edges connect precisely two unit squares sharing an edge. If the dominoes in R are weighted, then the edges in its dual graph G have the same weights as that of the corresponding dominoes. The tilings of a region R are in bijection with the perfect matchings of its dual graph G . In particular, $W(R) = W(G)$.

Kuo [Kuo04] proved the following combinatorial interpretations of the Dodgson condensation.

Theorem 3.3 (Theorem 5.1 in [Kuo04]). *Let $G = (V_1, V_2, E)$ be a (weighted) planar bipartite graph with $|V_1| = |V_2|$. Assume that u, v, w, s are four vertices appearing in a cyclic order on a face of G . Assume in addition that $u, w \in V_1$ and $v, s \in V_2$. Then*

$$W(G)W(G - \{u, v, w, s\}) = W(G - \{u, v\})W(G - \{w, s\}) + W(G - \{u, s\})W(G - \{v, w\}). \quad (3.3)$$

Theorem 3.4 (Theorem 5.2 in [Kuo04]). *Let $G = (V_1, V_2, E)$ be a planar bipartite graph with $|V_1| = |V_2| + 1$. Assume that u, v, w, s are four vertices appearing in a cyclic order on a face of G . Assume in addition that $u, v, w \in V_1$ and $s \in V_2$. Then*

$$W(G - \{v\})W(G - \{u, w, s\}) = W(G - \{u\})W(G - \{v, w, s\}) + W(G - \{w\})W(G - \{u, v, s\}). \quad (3.4)$$

Theorem 3.5 (Theorem 5.3 in [Kuo04]). *Let $G = (V_1, V_2, E)$ be a planar bipartite graph with $|V_1| = |V_2|$. Assume that u, v, w, s are four vertices appearing in a cyclic order on a face of G . Assume in addition that $u, v \in V_1$ and $w, s \in V_2$. Then*

$$W(G - \{u, s\})W(G - \{v, w\}) = W(G)W(G - \{u, v, w, s\}) + W(G - \{u, w\})W(G - \{v, s\}). \quad (3.5)$$

4 Proofs of the main results

We present only the proof of Theorem 2.3, as Theorem 2.4 can be treated by a completely analogous manner.

Proof of Theorem 2.3. We prove the equation (2.2) by induction on $k + s + t$. Recall that k is the cardinality of the index sets A and B , s is the number of contiguous components in B , and t is the sum to the sizes of the gaps in B . The base case is the case when $s = 1$, i.e. when $\text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ is a contiguous minor. This case follows directly from Theorem 1.1.

For the induction step, we assume that $s \geq 2$ and that (2.2) holds for any \mathcal{Q} -type regions in which the sum of their k -, s - and t -parameters strictly less than $k + s + t$.

We apply the Jaw Move in Lemma 3.2 to the $k \times (k + 1)$ matrix $M_A^{B \cup \{b+k-k_s+t-1\}}$ with $d = b + k + t - 1, e = b + k - k_s + t - 1, f = b, g = a$, and obtain

$$\begin{aligned} \text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1}) \text{SM}_{a+1,b+1}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1} - 1) = \\ \text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1} - 1) \text{SM}_{a+1,b+1}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1}) \\ + \text{SM}_{a,b+1}(k_1 - 1, \dots, k_s + 1; t_1, \dots, t_{s-1} - 1) \text{SM}_{a+1,b}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1}). \end{aligned} \quad (4.1)$$

Here we understand that

$$\text{SM}_{a,b}(k_1, \dots, k_{s-1}, 0; t_1, \dots, t_{s-1}) \equiv \text{SM}_{a,b}(k_1, \dots, k_{s-1}; t_1, \dots, t_{s-2}), \quad (4.2)$$

$$\text{SM}_{a,b}(0, k_2, \dots, k_s; t_1, \dots, t_{s-1}) \equiv \text{SM}_{a,b}(k_2, \dots, k_s; t_2, \dots, t_{s-1}), \quad (4.3)$$

and

$$\text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-2}, 0) \equiv \text{SM}_{a,b}(k_1, \dots, k_{s-2}, k_{s-1} + k_s; t_1, \dots, t_{s-2}). \quad (4.4)$$

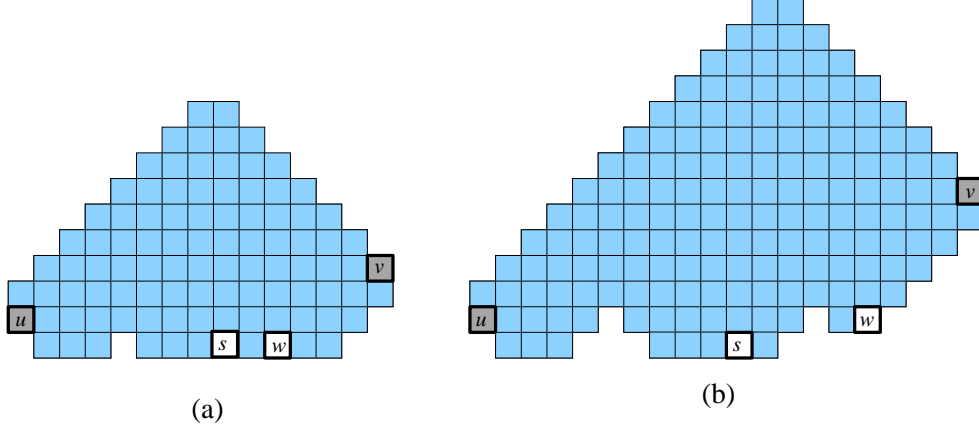


Figure 4.1: How we apply Kuo condensation in the case when $1 \leq t < h - k$.

To prove (2.2), we will use Kuo condensation to show that the tiling polynomial $P(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1}))$ satisfies the same recurrence. There are three cases to distinguish, based on the type of the region $\mathcal{Q} := \mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$.

To make sure our process runs smoothly, we assume by convention in the rest of this proof that

$$\mathcal{Q}_{x,h}(k_1, \dots, k_{s-1}, 0; t_1, \dots, t_{s-1}) \equiv \mathcal{Q}_{x,h}(k_1, \dots, k_{s-1}; t_1, \dots, t_{s-2}), \quad (4.5)$$

$$\mathcal{Q}_{x,h}(0, k_2, \dots, k_s; t_1, \dots, t_{s-1}) \equiv \mathcal{Q}_{x,h}(k_2, \dots, k_s; t_2, \dots, t_{s-1}), \quad (4.6)$$

$$\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-2}, 0) \equiv \mathcal{Q}_{x,h}(k_1, \dots, k_{s-2}, k_{s-1} + k_s; t_1, \dots, t_{s-2}). \quad (4.7)$$

Case 1. $t < h - k$.

We color the unit squares on the square lattice black and white so that two adjacent unit squares have different colors. Without loss of generality, we assume that the Aztec rectangle $\text{AR}_{x-h,0}^{h+k_1, h-k+k_1}$ (in the definition of Type 1 \mathcal{Q} -regions) has white unit squares along its northwest boundary.

By Remark 2.2, the top of the region $\mathcal{H} = \mathcal{H}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1})$ is always below or on the line $y = n$.

We apply Kuo's Theorem 3.5 to the dual graph of G the region $\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1})$. We pick the four vertices u, v, w, s as in Figure 4.1(a) (for $s = 3, k_1 = 2, k_2 = k_3 = 1, t_1 = 1, t_2 = 2, h = 8$ ³) and Figure 4.1(b) (for $s = 3, k_1 = 2, k_2 = 2, k_3 = 3, t_1 = 1, t_2 = 2, h = 12$). The vertices u, v correspond to the shaded unit squares, the vertices w, s correspond to the white unit squares with bold boundary in those figures. In particular, the vertices u and v correspond to the leftmost and the rightmost black unit square in the region; s corresponds to the white unit square at the position $2(k_2 + \dots + k_{s-1} + t_1 + \dots + t_{s-1})$ on the base, and w corresponds to the white unit square at the position $2(k_2 + \dots + k_s + t_1 + \dots + t_{s-1})$ on the base if such vertex exists (as in Figure 4.1(a)), otherwise we pick the w -square as the lowest white unit square on the stair going southwest from the unit square corresponding to v (as in Figure 4.1(b)).

³Our arguments in this proof work regardless the value of x . Thus, to make our illustrating figures simple, we do not give any particular value for x the figures.

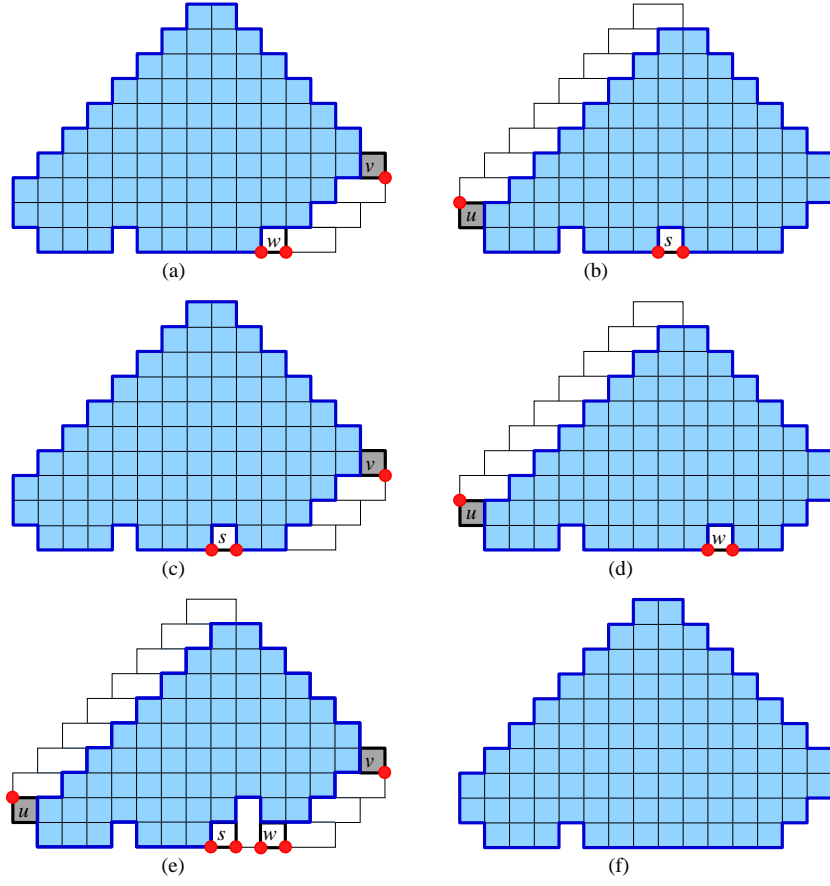


Figure 4.2: Obtaining the recurrence for $P(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1}))$ in case when $t \leq h - k$.

Consider the region corresponding to the graph $G - \{v, w\}$. It has several dominoes, which are forced to be in any tiling of the region. By removing these dominoes, we get back the region $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ (see the shaded region restricted by the bold contour in Figure 4.2(a) for $s = 3, k_1 = 2, k_2 = k_3 = 1, t_1 = 1, t_2 = 2, h = 8$).

Similarly, by removing forced dominoes from the regions corresponding to $G - \{u, s\}$, $G - \{v, s\}$, $G - \{u, w\}$, and $G - \{u, v, w, s\}$, we get the regions $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1} - 1)$, $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1} - 1)$, $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1})$ and $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s + 1; t_1, \dots, t_{s-1} - 1)$, respectively (see Figures 4.2(b)–(e)). Theorem 3.5 and Figure 4.2 tell us that the product of the weights of the two regions on the top is equal to the product of the weights of the two regions in the middle, plus the product of the weights of the two regions on the bottom. Equivalently, we have

$$\begin{aligned} C_1 C_2 W(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})) W(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1} - 1)) = \\ C_3 C_4 W(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1} - 1)) W(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1})) \\ + C_5 C_6 W(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s + 1; t_1, \dots, t_{s-1} - 1)) W(\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1})), \end{aligned} \quad (4.8)$$

where C_i is the products of weights of forced dominoes in the region corresponding to the i -th graph in the equation (3.5) in Theorem 3.5 (of course in this case, we have $C_6 = 1$).

Comparing the covering monomial of the region corresponding to G to the covering monomials of the other regions in (4.8), we obtain

$$\frac{F(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1}))}{F(\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1}))} = \frac{1}{C_1 D_v D_w}, \quad (4.9)$$

$$\frac{F(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1} - 1))}{F(\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1}))} = \frac{1}{C_2 D_u D_s}, \quad (4.10)$$

$$\frac{F(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1} - 1))}{F(\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1}))} = \frac{1}{C_3 D_v D_s}, \quad (4.11)$$

$$\frac{F(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1}))}{F(\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1}))} = \frac{1}{C_4 D_u D_w}, \quad (4.12)$$

$$\frac{F(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s + 1; t_1, \dots, t_{s-1} - 1))}{F(\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1}))} = \frac{1}{C_5 D_u D_v D_w D_s}, \quad (4.13)$$

where D_u (resp., D_v, D_w, D_s) is the product of those terms $v_{x,y}$, which correspond to the lattice points (x, y) adjacent the u -square (resp., v -square, w -square, s -square), and are not the central of the long side of any forced domino (illustrated by the red dots in Figures 4.2(a)–(e)).

By (4.8)–(4.13), we get

$$\begin{aligned} P(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})) P(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1} - 1)) = \\ P(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1} - 1)) P(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1})) \\ + P(\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s + 1; t_1, \dots, t_{s-1} - 1)) P(\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1})). \end{aligned} \quad (4.14)$$

This means that $\text{SM}_{a,b}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ and $P(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1}))$ satisfy the same recurrence in this case.

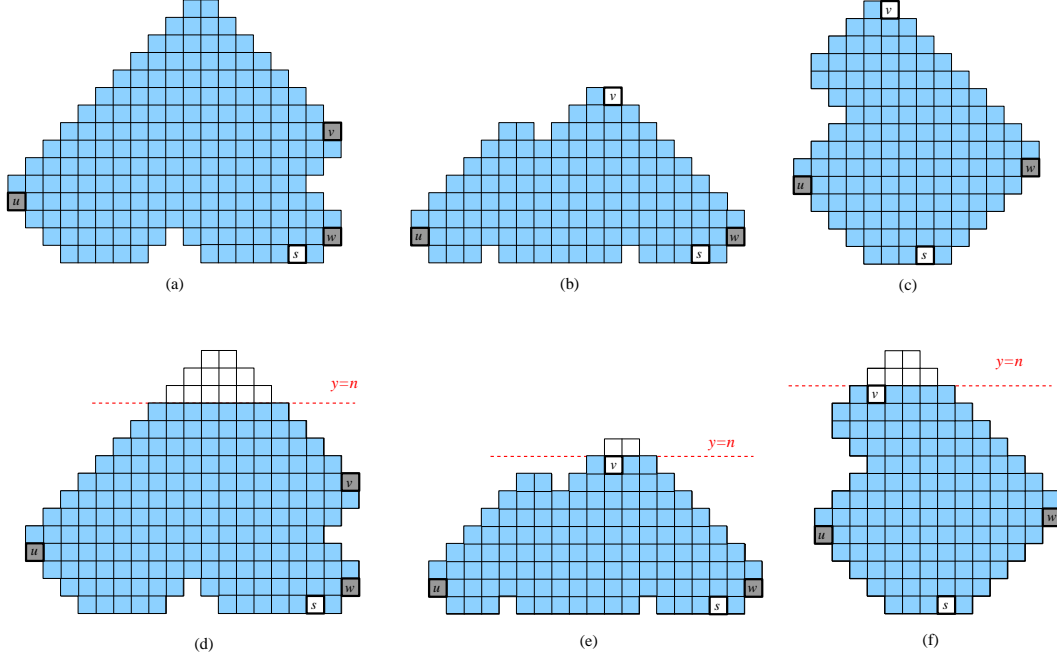


Figure 4.3: How we apply Kuo condensation when $h \geq 0^+$ and $t > h - k$.

Case 2. $h \geq 0^+$ and $t \geq h - k$.

If $t = h - k$, then the region $\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1})$ is still of Type 1; and the process in Case 1 still works. Thus, we can assume that $t > h - k$. We work first to the case the top of \mathcal{H} is below or on the line $y = n$.

We consider first the subcase when $1 \leq t + k - h \leq k_1 - 1$. This corresponds to the case AD_2 stays inside AD_1 . We apply Kuo's Theorem 3.4 to the graph G of the region R that is the union of the regions $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ and $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1})$ as in Figure 4.3(a) for $s = 3, k_1 = 4, k_2 = k_3 = 2, t_1 = 2, t_2 = 3, h = 11$. We color the unit squares of R like a chessboard, so that the unit squares on the northwest side of AD_1 are white. The vertex u corresponds to the leftmost black unit square in R , v corresponds to the last black unit square on the stair going southeast from the top, w corresponds to the leftmost black unit square on the stair going northeast from the bottom, and the unit square corresponding to s is the leftmost white unit square on the base. We obtain the regions $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$, $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1})$, $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1} - 1)$, $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1} - 1)$, and $\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1})$ by removing forced dominoes from the regions corresponding to the graphs $G - \{v\}$, $G - \{u, w, s\}$, $G - \{v, w, s\}$, $G - \{u\}$, $G - \{u, v, s\}$ and $G - \{w\}$, respectively (see Figures 4.4(a)–(f) respectively). We get again the equation (4.8) as in Case 1, where C_i is now the product of weights of forced dominoes in the region corresponding to the i -th graph in the equation (3.4) of Theorem 3.4. By comparing the covering the monomials of the regions in (4.8) to the covering monomial of R , we get also (4.14).

Next, we investigate the subcase when $t + k - h \geq k_1$, then AD_2 does not stay inside AD_1 any more. In this case, Theorem 3.3 has been used for the dual graph of G of $\mathcal{Q} :=$

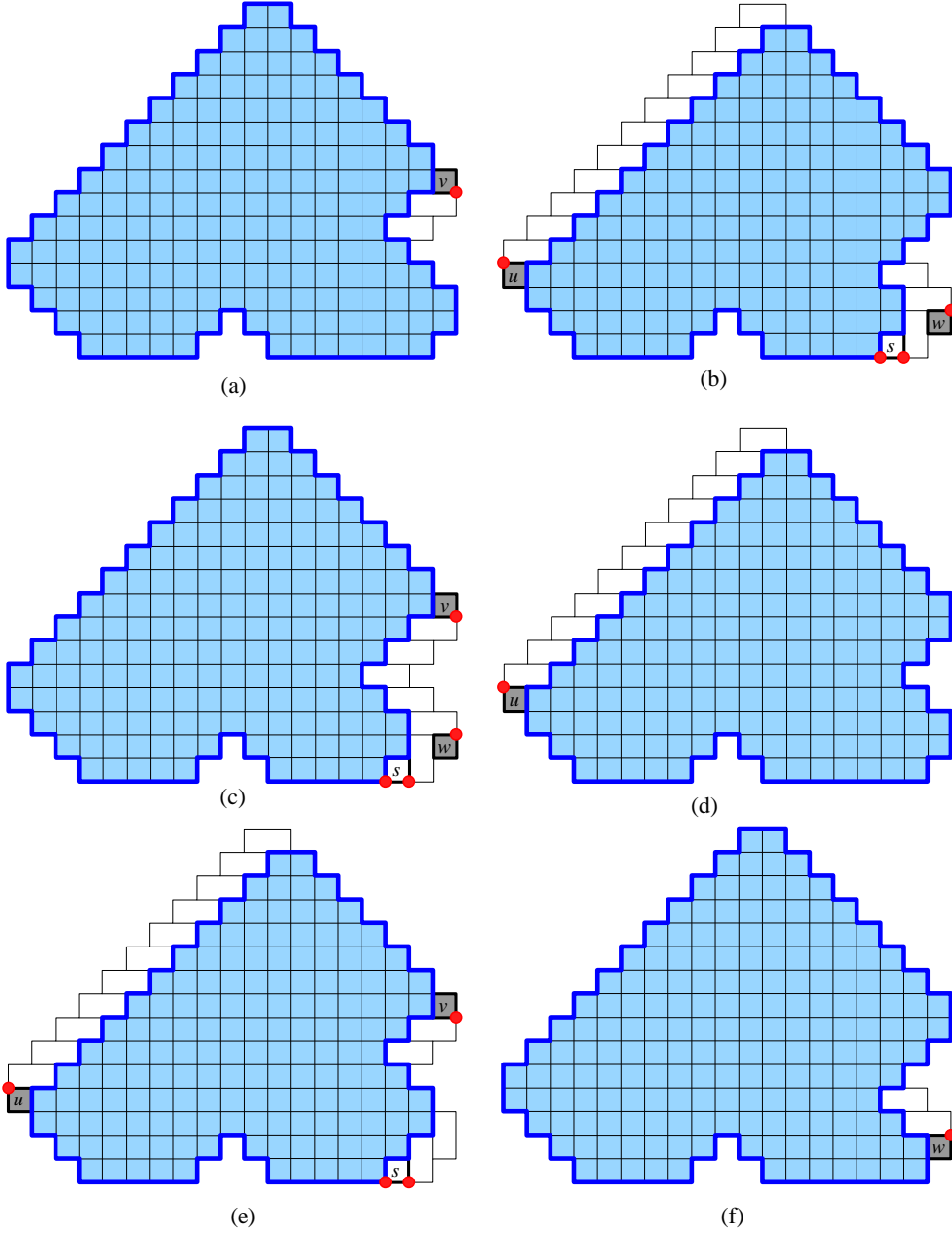


Figure 4.4: Obtaining the recurrence for $P(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1}))$ in case when $1 \leq t + k - h \leq k_1 - 1$.

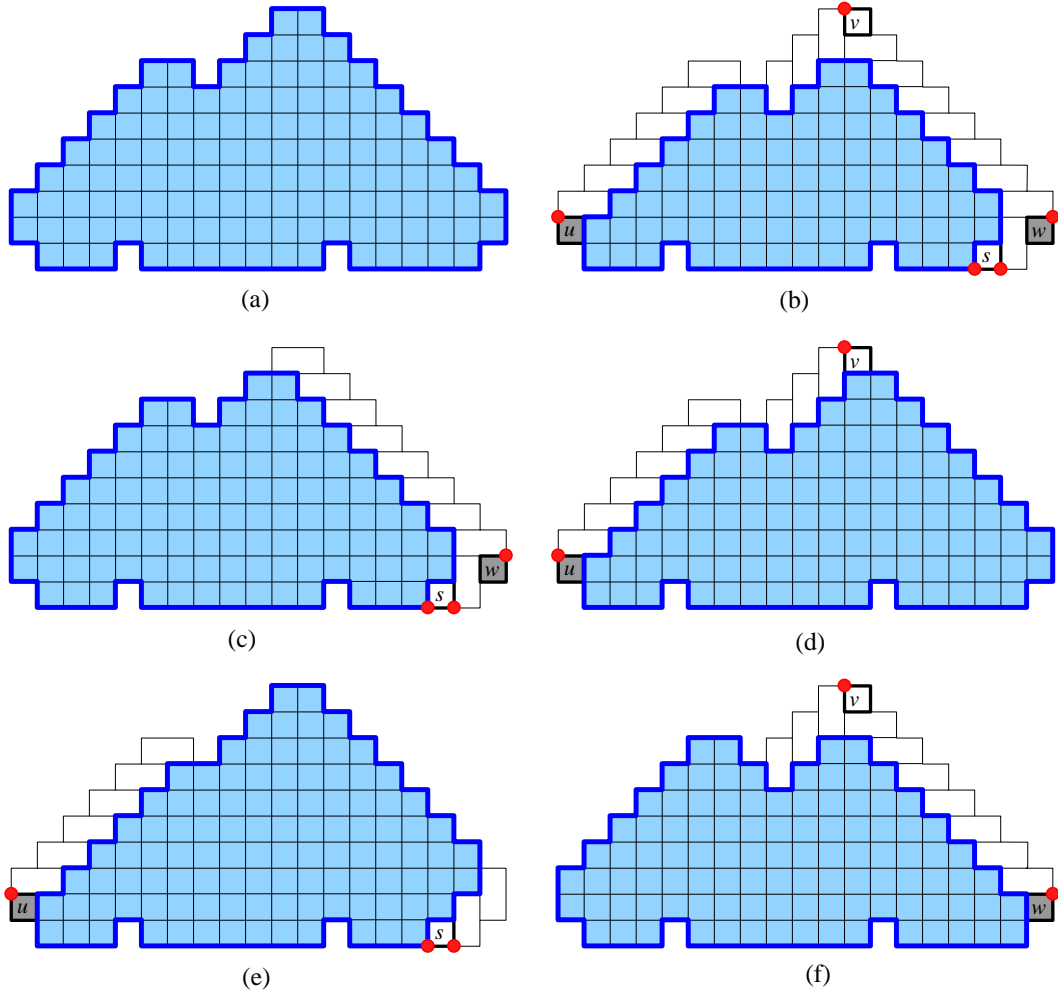


Figure 4.5: Obtaining the recurrence for $P(\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1}))$ in case when $t+k-h \geq k_1$.

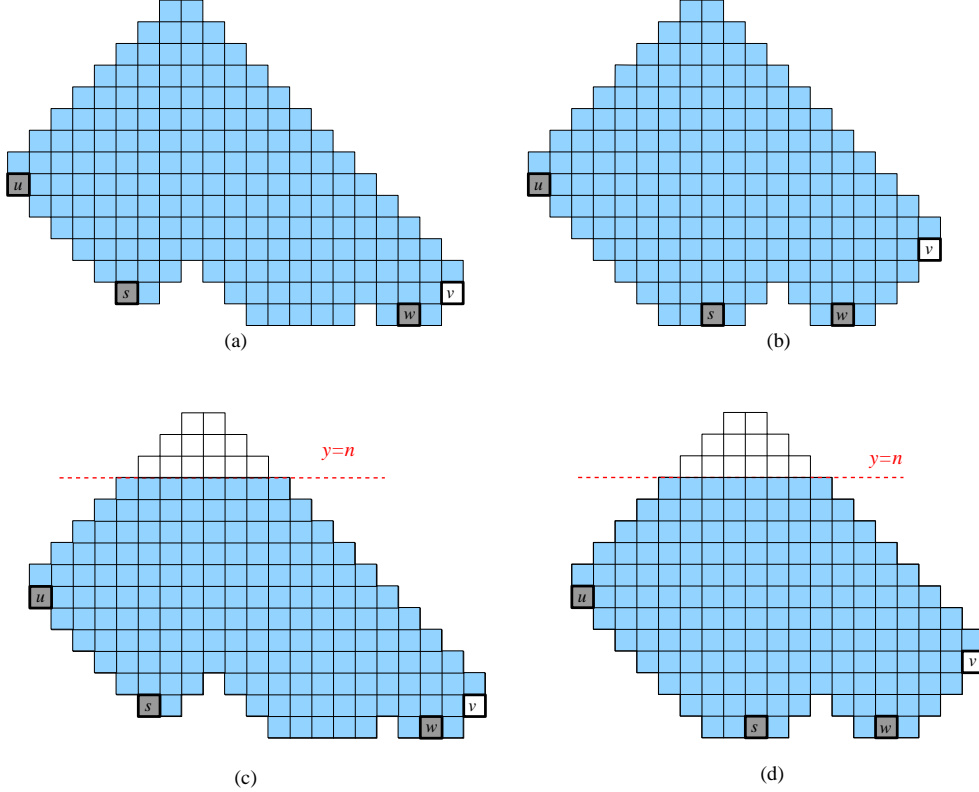


Figure 4.6: How we apply Kuo condensation when $h \leq 0^-$.

$\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ as in Figure 4.3(b) for $s = 4, k_1 = 2, k_2 = k_3 = 1, k_4 = 2, t_1 = 1, t_2 = 3, t_3 = 2, h = 6$ and Figure 4.3(c) for $s = 2, k_1 = 5, k_2 = 6, t_1 = 2, h = 3$. More precise, the u -square and w -square are the leftmost and the rightmost black unit squares of \mathcal{Q} , the w -square is the white unit square on the top of AD_2 , and the s -square is still the leftmost white unit square on the base. By removing the forced dominoes, we respectively transform the regions corresponding to the graphs $G - \{u, v, w, s\}$, $G - \{w, s\}$, $G - \{u, v\}$, $G - \{u, s\}$ and $G - \{v, w\}$ into the regions in the equation (4.8) (see Figures 4.5 (b)–(f) respectively). By the same process as in the previous cases, we also obtain (4.14).

We now consider the situation when the top of \mathcal{H} is above the line $y = n$. It is easy to see that when $1 \leq t + k - h \leq k_1 - 1$, our region looks like Figure 4.3(d). Then the Kuo's condensation still works here the same as in Figures 4.3(a) and 4.4. In the case when $k - h \geq k_1$ (i.e. AD_1 stays inside AD_2), our region looks like 4.3(f); if $t + k - h \geq k_1 > k - h$ (i.e. AD_1 and AD_2 are not inside each other), our region has the structure as in Figure 4.3(e). In this case we pick u, w, s similarly as in Figures 4.3(b) and (c), however, the v -square is now the second square of the truncated top of AD_2 if it is cut off by the line $y = n$. Then Kuo condensation still works the same as in Figure 4.5.

Case 3. $h \leq 0^-$.

Similar to the above cases, we assume that the square lattice has a chessboard coloring, so that the Aztec rectangle $AR_{x-h+t,0}^{k+t-h-k_1, 2k+t-h-k_1}$ (in the definition of the type-3 \mathcal{Q} regions) has

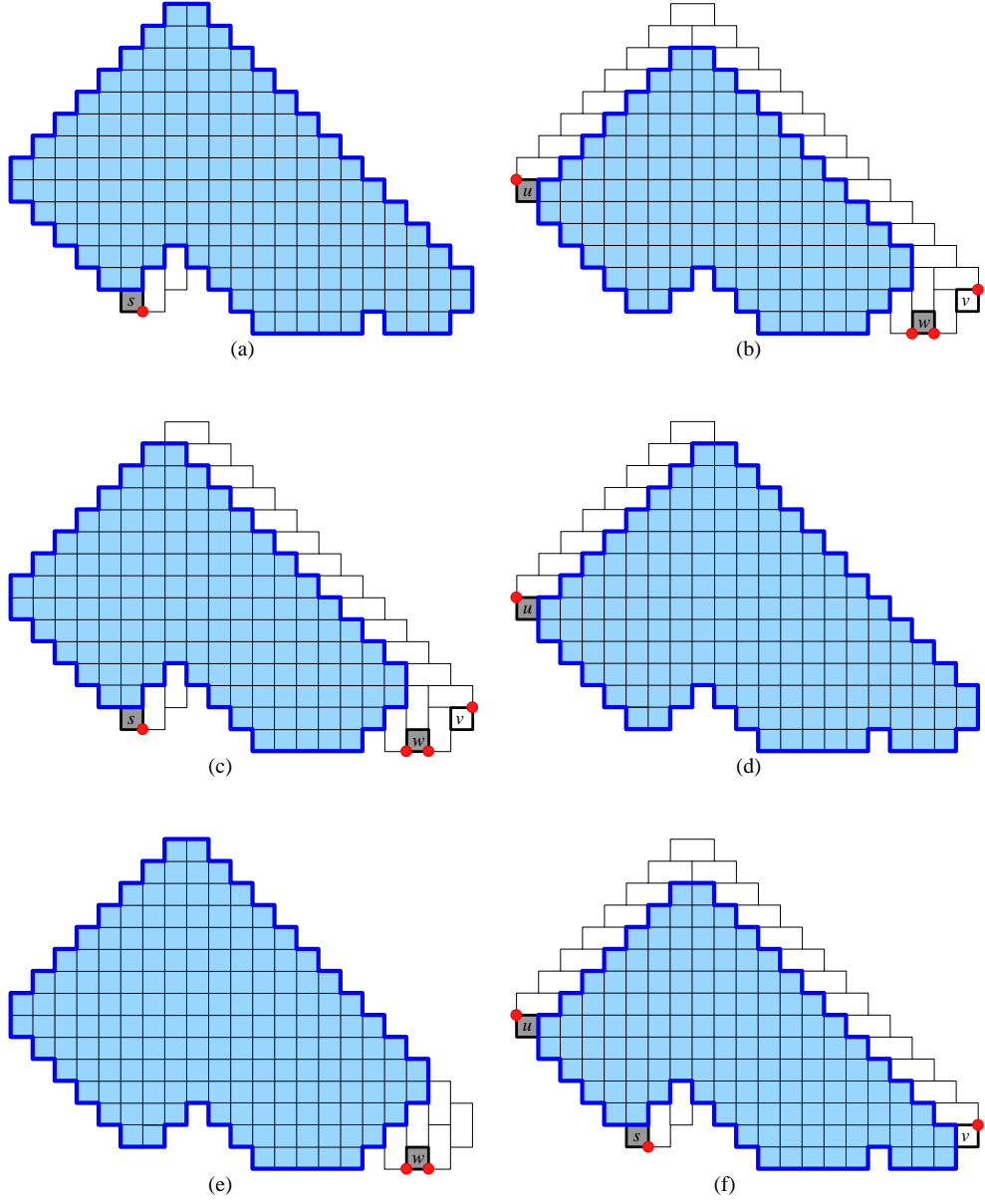


Figure 4.7: Obtaining the recurrence when $h \leq 0^-$.

white unit squares on the northwest boundary.

We assume first that the height of \mathcal{H} not greater than n . We apply Kuo's Theorem 3.4 to the dual graph G of the union R of the two regions $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$ and $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1})$ as in Figure 4.6(a) (for $s = 3, k_1 = 4, k_2 = 1, k_3 = 2, t_1 = 2, t_2 = 1, h = -2$) and Figure 4.6(b) (for $s = 2, k_1 = 3, k_2 = 4, t_1 = 1, h = -3$). In this case, the u -square is the leftmost black unit square in R , the v -square is the rightmost white unit square, the w -square is the leftmost black unit square on the base, and the s -square is the black unit square at the position $2(k_1 + \dots + k_{s-1} + t_1 + \dots + t_{s-1})$ (from the right) on the base if such vertex exists (as in Figure 4.6(b)), otherwise we pick it as the last black unit square on the stair going southeast from the u -square (as in Figure 4.6(a)). By Figures 4.7(a)–(f), we get the regions $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1})$, $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1} - 1)$, $\mathcal{Q}_{x,h}(k_1, \dots, k_s; t_1, \dots, t_{s-1} - 1)$, $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s; t_1, \dots, t_{s-1})$, $\mathcal{Q}_{x+1,h}(k_1 - 1, \dots, k_s + 1; t_1, \dots, t_{s-1} - 1)$, and $\mathcal{Q}_{x,h}(k_1, \dots, k_s - 1; t_1, \dots, t_{s-1})$ by removing forced dominoes from the regions corresponding to the graphs $G - \{s\}$, $G - \{u, v, w\}$, $G - \{v, w, s\}$, $G - \{u\}$, $G - \{w\}$, and $G - \{u, v, s\}$, respectively. Then (4.14) follows from Theorem 3.4 in the same way as in the previous cases.

Finally, if the top of \mathcal{H} is above the line $y = n$, i.e. the top of $\text{AR}_{x-h+t,0}^{k+t-h-k_1, 2k+t-h-k_1}$ is above the line $y = n$. Then we apply Kuo condensation in the same way as in the above case (illustrated by Figures 4.6(c) and (d)). This finishes our proof. \square

5 Open question for general circular minors

We would like to know when the statement of Kenyon-Wilson Conjecture 1.2 holds for the circular minors $\det M_A^B$ that have *both* index sets A and B non-contiguous. Equivalently, we want to characterize the circular minors that can be written as the tiling polynomial $P(R)$ of a region R on the square lattice.

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